From the expansion (3.6) it is seen that in the case of potential mass forces, some time after the start of the flow, regardless of the initial condition, the lowest harmonic will predominate in it,

$$\sum_{m=-1}^{1} a_{11}^{m} \exp\left(-\lambda_{11}^{-} t\right) \mathbf{v}_{11}^{m-};$$

it represents differential rotation about some axis proportional to the aximuthal component of a helical Hill vortex, described in [5], and damped in proportion to exp $(-20.19\nu R^{-2}t)$.

LITERATURE CITED

1. A. N. Tikhonov and A. A. Samarskii, Equations of Mathematical Physics [in Russian], Nauka, Moscow (1972).
2. N. S. Koshlyakov, É. B. Gliner, and M. M. Smirnov, Equations in Partial Derivatives of Mathematical Physics [in Russian], Vysshaya Shkola, Moscow (1970).
3. V. M. Bykov, "Helical flows inside a sphere," Zh. Prikl. Mekh. Tekh. Fiz., No. 2 (1979).
4. O. A. Ladyzhenskaya, Mathematical Problems of the Dynamics of a Viscous Incompressible Fluid [in Russian], Nauka, Moscow (1970).
5. A. G. Yarmitskii, "On one three-dimensional analog of a Chaplygin vortex column (generalized Hill vortex)," Zh. Prikl. Mekh. Tekh. Fiz., No. 5 (1974).

NONISOTHERMAL FLOW INDUCED BY THE SQUEEZING OF A NON-NEWTONIAN FLUID FILM BETWEEN TWO PARALLEL PLATES

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UDC 532.5:532.135

In the pressure forming of thin-sheet products, the polymer melt is injected into the cavity formed by partially contacting half-molds. The next stage is the joining of the half-molds, during which time the melt is squeezed to fill the mold cavity and harden at the end of the process.

Here we consider the problem of nonisothermal flow induced in a molten polymer film between two parallel plates (half-molds), which are squeezed together at a rate v in the direction normal to the plane of the plates. We investigate the temperature regime of the fluid cooling process as a function of the governing parameters of the problem.

It is assumed that the fluid is incompressible and obeys a power rheological law, where the consistency depends on the temperature T: $\mu = \mu(T)$.

The temperature of the fluid at the initial time is T_0, and the wall temperature is T_W ($T_W \ll T_0$).

To the best of our knowledge, this kind of problem has been investigated only in [1]. However, to simplify the solution the authors have, without justification, rejected the convection term in the heat-balance equation.

We introduce a cylindrical coordinate system with the z axis directed perpendicular to the plane of the plates and with the origin situated at the center of the lower plate (Fig. 1). The radius of the fluid film R(t) is a function of the time t and is determined from the condition of a constant initial volume of the fluid.

Taking axial symmetry into account, we find that the tangential component of the velocity v_φ and the derivatives of all variables with respect to φ are equal to zero.

Under the condition that body forces and surface-tension forces are negligible, the stated problem corresponds to the system of equations

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 70-75, March-April, 1980. Original article submitted March 29, 1979.

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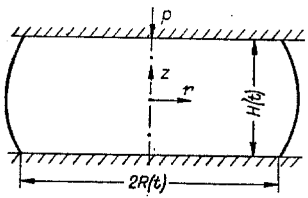

Fig. 1

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+v_z\frac{\partial v_r}{\partial z}\right)=-\frac{\partial p}{\partial r}+\frac{\partial}{\partial r}\left[2\mu\left(\frac{\partial v_r}{\partial r}\right)^n\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v_z}{\partial r}+\frac{\partial v_r}{\partial z}\right)^n\right]+\frac{2}{r}\mu\left[\left(\frac{\partial v_r}{\partial r}\right)^n-\left(\frac{v_r}{r}\right)^n\right]; \tag{1}$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\frac{\partial}{\partial r}\left[\mu\left(\frac{\partial v_z}{\partial r}+\frac{\partial v_r}{\partial z}\right)^n\right]+\frac{\partial}{\partial z}\left[2\mu\left(\frac{\partial v_z}{\partial z}\right)^n\right]+\frac{1}{r}\mu\left(\frac{\partial v_z}{\partial r}+\frac{\partial v_r}{\partial z}\right)^n; \tag{2}$$

$$\frac{\partial T}{\partial t}+v_r\frac{\partial T}{\partial r}+v_z\frac{\partial T}{\partial z}=a\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\partial^2 T}{\partial z^2}\right]; \tag{3}$$

$$\frac{\partial (rv_r)}{\partial r}+\frac{\partial (rv_z)}{\partial z}=0, \tag{4}$$

in which r is the radial coordinate, v_r and v_z are the radial and transverse components of the velocity, ρ is the density of the fluid, p is the pressure, n is the exponent in the power rheological law, and a is the thermal diffusivity.

Considerable difficulties are met in the solution of the system (1)-(4), even by computer-oriented numerical methods. We use the following procedure to simplify the system (1)-(4): By a suitable choice of scales we transform the system to dimensionless form in such a way as to make the order of each variable equal to unity [2]. Under this transformation the influence of the terms of the equations are estimated by the dimensionless factor associated with the transformation from the system (1)-(4) to the system containing dimensionless variables.

We introduce the scales R(t) for the radial coordinate; H(t)/2 for the transverse coordinate, where H(t) is the distance between the plates; $v_0 = -H(0)f'(0)$ for the transverse velocity, where $f(t) = H(t)/H(0)$; $(T_0 - T_W)$ for the temperature; $\tau_0 = H(0)/2v_0$ for the time.

The scale for the radial component of the velocity, denoted by v_1, is obtained from the equation of continuity (4) and is expressed in terms of the above-defined velocity and length scales in the form $v_1 = 2R(t)v_0/H(t)$.

We take as the characteristic pressure p_0 the average pressure on the plates for the case of isothermal flow of the fluid; this quantity can be estimated by analogy with the approach in [3], where isothermal flow induced by the squeezing of a Newtonian fluid layer between two parallel plates is investigated.

The heat-conduction equation must be transformed with regard for the existence of different time scales characterizing the dynamics of evolution of the fluid flow and the temperature field. If we assume that the temperature variation is determined mainly by molecular heat conduction, then by replacing the derivatives in Eq. (3) (in which convection terms are neglected on the left-hand side) by the ratios of finite increments, we obtain a time scale characteristic of the evolution of the temperature field: $\tau_1 = H^2(0)/4a$.

The transformation from the system (1)-(4) to the system containing dimensionless variables and the subsequent order-of-magnitude estimation of the terms with regard for the fact that $H(t) \ll R(t)$ and $T_W \ll T_0$ yield major simplifications. Omitting the simple calculations, we give the final form of the approximate system of equations, rewriting it in the new dimensionless variables:

$$\left(-\frac{dP}{d\eta}\right)f^n(\tau)=-\frac{\partial}{\partial\xi}\left[\Psi(\Theta)\left(\frac{\partial V_\eta}{\partial\xi}\right)^n\right]; \tag{5}$$

$$\frac{\partial\Theta}{\partial\tau}+\mathrm{Pe}\,f(\tau)\,V_\xi\frac{\partial\Theta}{\partial\xi}=\frac{\partial^2\Theta}{\partial\xi^2}; \tag{6}$$

$$\frac{1}{\eta}\frac{\partial(\eta V_\eta)}{\partial\eta}+\frac{\partial V_\xi}{\partial\xi}=0, \tag{7}$$

where $\Theta = (T - T_w)/(T_0 - T_w)$; $\eta = 2r/H(t)$; $\xi = 2z/H(t)$; $V_\xi = v_z/v_0$, $V_\eta = v_r/v_0$; $P = p/P_1$, $P_1 = \mu(T_0)(2v_0/H(0))^n$; $\tau = \int\limits_0^t 4a dt_1/H^2(t_1)$; $\Psi(\Theta) = \mu(T)/\mu(T_0)$; and $\mathrm{Pe} = v_0 H(0)/2a$ is the Peclet number.

The system (5)-(7) is solved under the following initial and boundary conditions:

$$V_\eta = 0,\ V_\xi = \varphi(\tau),\ \Theta = 0 \ \text{for} \ \xi = 0; \tag{8}$$

$$\partial V_\eta/\partial\xi = 0,\ V_\xi = 0,\ \partial\Theta/\partial\xi = 0 \ \text{for} \ \xi = 1; \tag{9}$$

$$\Theta = 1\ (0 < \xi \leqslant 1),\ \Theta = 0 \ \text{for} \ \xi = 0,\ \tau = 0; \tag{10}$$

$$P = 0 \ \text{for} \ \eta = R_1(\tau), \tag{11}$$

where the radius of the film $R_1(\tau)$ is found from the condition of a constant fluid volume:

$$R_1(\tau) = 2R(0)f^{-3/2}(\tau)/H(0),\ \varphi(\tau) = f'(\tau)/f'(0),\ f(\tau) = H(\tau)/H(0).$$

Simple transformations reduce the solution of the system (5)-(11) to the solution of a nonlinear integro-differential equation of the heat-conduction type.

Thus, integrating Eq. (5) subject to the first boundary condition (9), we obtain

$$\frac{\partial V_\eta}{\partial\xi} = \left(-\frac{dP}{d\eta}\right)^{1/n} f(\tau)\left[(1-\xi)\Psi^{-1}(\Theta)\right]^{1/n}.$$

Integrating the resulting equation from 0 to ξ and invoking the first boundary condition (8), we obtain an expression for the radial velocity:

$$V_\eta = (-dP/d\eta)^{1/n} f(\tau)\Phi(\tau, \xi), \tag{12}$$

where

$$\Phi(\tau, \xi) = \int\limits_0^\xi \left[(1-\xi_1)\Psi^{-1}(\Theta)\right]^{1/n} d\xi_1.$$

From the equation of continuity (7), using the boundary conditions (8) and (9), we obtain

$$\eta\varphi(\tau)\, d\eta = \left(\frac{\partial}{\partial\eta}\int\limits_0^1 \eta V_\eta d\xi\right) d\eta. \tag{13}$$

Integrating Eq. (13) from 0 to η, we have

$$\varphi(\tau)\eta^2/2 = \int\limits_0^1 \eta V_\eta d\xi + \text{const}.$$

Setting $\eta = 0$ in this equation, we find in succession

$$\text{const} = 0,$$

$$\varphi(\tau)\eta^2/2 = \eta\left(-\frac{dP}{d\eta}\right)^{1/n} f(\tau)\int\limits_0^1 \Phi(\tau, \xi)\, d\xi. \tag{14}$$

We introduce the function

$$\Phi_1(\tau) = \left\{\int\limits_0^1 \Phi(\tau, \xi)\, d\xi\right\}^{-n} = \left\{\int\limits_0^1 d\xi \int\limits_0^\xi \left[(1-\xi_1)\Psi^{-1}(\Theta)\right]^{1/n} d\xi_1\right\}^{-n}$$

and rewrite Eq. (14) in the form

$$\left(-\frac{dP}{d\eta}\right) = \left[\varphi(\tau)/f(\tau)\right]^n (\eta/2)^n \Phi_1(\tau). \tag{15}$$

We are now in a position to eliminate the factor $(-dP/d\eta)$ in (12):

$$V_\eta = \varphi(\tau)\eta\Phi(\tau, \xi)\Big/\left(2\int\limits_0^1 \Phi(\tau, \xi)\, d\xi\right). \tag{16}$$

Integrating the pressure-gradient equation (15) from η to $R_1(\tau)$ and taking (11) into account, we obtain

$$P(\eta, \tau) = \frac{1}{n+1}\left[\frac{\varphi(\tau)}{f(\tau)}\right]^n \frac{1}{\left(2\int_0^1 \Phi(\tau, \xi)\, d\xi\right)^n} \left(R_1^{n+1}(\tau) - \eta^{n+1}\right),$$

from which we determine the average pressure on the plates

$$\langle P\rangle = 2\int_0^{R_1(\tau)} P(\tau, \eta)\, d\eta / R_1^2(\tau) = \frac{1}{n+3}\left[\frac{\varphi(\tau)}{f(\tau)}\right]^n \left[\frac{2R_1(0)}{H(0)}\right]^{n+1} f^{-3(n+1)/2}(\tau) \Bigg/ \left(2\int_0^1 \Phi(\tau, \xi)\, d\xi\right)^n. \tag{17}$$

Returning to Eq. (7), we rewrite it with regard for (16):

$$\frac{\partial V_\xi}{\partial \xi} = -\frac{1}{\eta}\frac{\partial(\eta V_\eta)}{\partial \eta} = -\varphi(\tau)\Phi(\tau, \xi) \Bigg/ \int_0^1 \Phi(\tau, \xi)\, d\xi. \tag{18}$$

We integrate Eq. (18) from 1 to ξ. The second boundary condition (9) enables us to obtain an explicit expression for the transverse velocity component:

$$V_\xi = \varphi(\tau)\int_\xi^1 \Phi(\tau, \xi)\, d\xi \Big/ \int_0^1 \Phi(\tau, \xi)\, d\xi.$$

We have thus reduced the problem to the solution of the integrodifferential equation

$$\frac{\partial \Theta}{\partial \tau} + \mathrm{Pe}\, f(\tau)\, \varphi(\tau)\left[\int_\xi^1 \Phi(\tau, \xi)\, d\xi \Big/ \int_0^1 \Phi(\tau, \xi)\, d\xi\right]\frac{\partial \Theta}{\partial \xi} = \frac{\partial^2 \Theta}{\partial \xi^2} \tag{19}$$

subject to the boundary and initial conditions

$$\Theta = 0 \text{ for } \xi = 0; \tag{20}$$

$$\partial\Theta/\partial\xi = 0 \text{ for } \xi = 1; \tag{21}$$

$$\Theta = 1\ (0 < \xi \leqslant 1,\ \tau = 0),\ \Theta = 0 \text{ for } \xi = 0. \tag{22}$$

For Pe = 0 (i.e., in the absence of thermal convection) the system (19)-(22) admits an exact solution [4]. The system (19)-(22) is approximated by a six-point implicit difference scheme and solved numerically on a computer.

This approach has been used previously for the numerical solution of the system of boundary-layer equations in a compressible gas flowing longitudinally past a plate [5].

It is assumed in all the calculations that the temperature dependence of the reciprocal of the fluid consistency has the form

$$\Psi^{-1}(\Theta) = \begin{cases} 1/\exp\,[b/(\Theta - \Theta_1)], & \text{if } \Theta > \Theta_1, \\ 0, & \text{if } \Theta \leqslant \Theta_1, \end{cases} \tag{23}$$

where b and Θ_1 are positive constants, and the velocity of the plates is constant, $\varphi(\tau) \equiv 1$; in this case, as is readily verified,

$$f(\tau) = 1/(1 + \mathrm{Pe}\tau).$$

Figure 2 illustrates the evolution of the temperature field with time (n = 0.33, b = 1.35, Θ_1 = 0.2). The parameter indicated by the numbers attached to the solid curves is the dimensionless time (Pe = 40 for all the

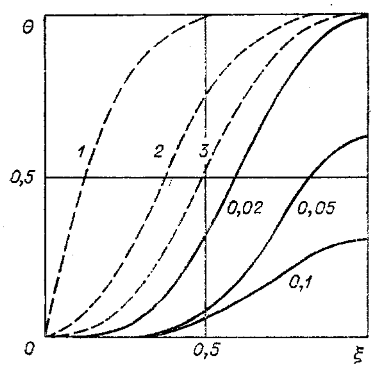

Fig. 2

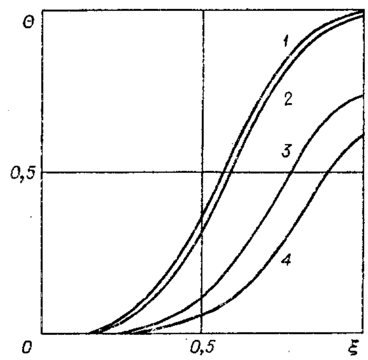

Fig. 3

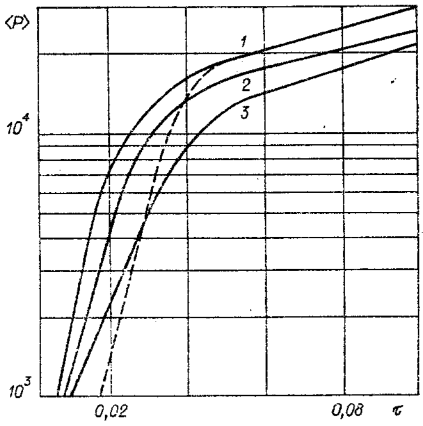

Fig. 4

curves). The dashed curves represent the temperature distribution for fixed $\tau = 0.02$ and various values of the Péclet number: 1) Pe = 0; 2) Pe = 20; 3) Pe = 30.

The error of the calculations can be estimated indirectly by comparing curve 1 with the exact solution in [4]. This comparison discloses their full agreement up to the third decimal place, i.e., within the error limits of the computational scheme (the computational steps are $\Delta\xi = 0.01$, $\Delta\tau = 0.001$, and the order of approximation of the difference scheme is $o(\Delta\tau + \Delta\xi^2)$ [5]).

The temperature distribution of the fluid as a function of the parameters n and b is given in Fig. 3 (Pe = 40, $\Theta_1 = 0.2$): 1) $\tau = 0.02$, b = 1.35, n = 1; 2) $\tau = 0.02$, b = 1.35, n = 0.5; 3) $\tau = 0.04$, n = 0.33, b = 1.35; 4) $\tau = 0.04$, n = 0.33, b = 0.5.

A graphic representation of the most characteristic features of the variation of the average pressure on the plates is given in Fig. 4, in which the solid curves correspond to fixed values of the parameters n = 0.33, b = 1.35, $\Theta_1 = 0.2$, $\beta = 2R_1(0)/H(0) = 50$. The parameter of the curves is the Péclet number: 1) Pe = 40; 2) Pe = 30; 3) Pe = 20. The dashed curve represents the variation of the average pressure on the plates for b = 0.5 (the other parameters have the same values as for curve 1).

As expected, the curves $\langle P \rangle = F(\tau)$ must tend to the same asymptote as $\tau \to \infty$, irrespective of the exponent b of the exponential function in (23), and so the differences between them (for a fixed value of Pe) become inconsequential, beginning with a certain value of τ.

LITERATURE CITED

1. S. Z. Mikhailov, Yu. V. Kazankov, A. A. Malinin, and K. A. Salazkin, "Filling of the dies in the injection pressure-forming regime," Tr. Mosk. Inst. Khim. Mashinostr., No. 54 (1974).
2. S. J. Kline, Similitude and Approximation Theory, McGraw-Hill, New York (1965).
3. S. M. Targ, Fundamental Problems of Laminar Flow Theory [in Russian], GITTL, Moscow–Leningrad (1951).
4. A. V. Lykov, Theory of Heat Conduction [in Russian], Vyssh. Shkola, Moscow (1967).
5. I. Yu. Brailovskaya and L. A. Chudov, "Solution of boundary-layer equations by the difference method," in: Computational Methods and Programming [in Russian], Mosk. Univ., Moscow (1962).